Proving Explicit Statements

Problem-1. Let d, k, q be positive integers and k being odd , find greatest power of 2 , in the decomposition of $\sum_{n=1}^{2^d k} n^q$.

AMM

Solution: we prove that if , being an even number or $d = 1$ the answer is $d - 1$ otherwise the anser is $2(d-1)$. if $q = 1$ it is obvious , if $d = 1$ the resulting sum is odd , now assume that $d > 1$ we can write the sum as the following:

$$(2^d k)^q + (2^{d-1}k)^q + \sum_{i=0}^{2^{d-1}k}(i^q + (2^d k - i)^q)$$

For even , take the sum modulo 2^d , thus we find that $\sum_{n=1}^{2^d k} n^q \equiv 2\sum_{n=1}^{2^{d-1}k} n^q \pmod{2^d}$, thus by use of induction on d we find that $\sum_{n=1}^{2^{d-1}k} n^q$ is congruent to 2^{d-2} modulo 2^{d-1} so $\sum_{n=1}^{2^d k} n^q \equiv 2^{d-1} \pmod{2^d}$ and it is the desired result.

For odd , notice that by use of binomial theorem , $i^q + (2^d k - i)^q \equiv 2^d qkn^{q-1} \pmod{2^{2d-1}}$, so $\sum_{n=1}^{2^d k} n^q \equiv 2^d qk \sum_{n=1}^{2^{d-1}k} n^{q-1} \pmod{2^{2d-1}}$ since $q-1$ is even by use of the former part we find that $\sum_{n=1}^{2^{d-1}k} n^{q-1} \equiv 2^{d-2} \pmod{2^{d-1}}$ so $\sum_{n=1}^{2^d k} n^q \equiv 2^{2d-2} \pmod{2^{2d-1}}$. and our proof is complete.

Variant :

Prove that $1^n + 2^n + \cdots . + (2^k - 1)^n$ is divisible by 2^k. **Kvant-M2277**

Solution: lets name the sum as , if n is odd , then rewrite A as the form $(1^n + (2^k - 1)^n) + (2^n + (2^k - 2)^n) + \cdots + ((2^{k-1} - 1)^n + (2^{k-1} + 1)^n) + 2^{n(k-1)}$ so it is obvious that A is divisible by 2^k .

If n is even , we made an induction on k assume the statement holds true for k then we must prove it for $k + 1$implies that , we know that $a^n \equiv (2^{k+1} - a)^n \pmod{2^{k+1}}$ thus:

$$1^n + 2^n + \cdots . + (2^{k+1} - 1)^n = \quad 1^n + (2^{k+1} - 1)^n) + (2^n + (2^{k+1} - 2)^n) + \cdots + ((2^{k+1} - 1)^n + (2^{k+1} + 1)^n) + 2^{kn} \equiv 2(1^n + 2^n + \cdots . + (2^k - 1)^n) + 2^{kn} \pmod{2^{k+1}}$$

By induction hypothesis $(1^n + 2^n + \cdots . + (2^k - 1)^n)$ is divisible by 2^k ,thus $1^n + 2^n + \cdots . + (2^{k+1} - 1)^n$ must being divisible by2^{k+1} so our proof is complete .

Problem-2. Prove that $1 + 3^3 + 5^5 + \cdots . + (2^n - 1)^{2^n - 1} \equiv 2^n \pmod{2^{n+1}}$ **Kvant-M2252**

Solution: at first we use two facts:

Fact-1: for odd , $k^{2^n} \equiv 1 \pmod{2^{n+2}}$

Fact-2: $(k + 2^n)^k \equiv k^k(1 + 2^n) \pmod{2^{n+2}}$

The former , could be proven by the identity $k^{2^n}-1=(k-1)(k+1)(k^2+1)\dots.\left(k^{2^{n-1}}+1\right)$ or by induction on n , and the later could be proven by use of binomial theorem. Lets define $S_n=1+3^3+5^5+\cdots.+(2^n-1)^{2^n-1}$, then $S_{n+1}=S_n+R_n$, where $R_n=\ (2^n+1)^{2^n+1}+\cdots.+(2^{n+1}-1)^{2^{n+1}-1}$, all the 2^n-1 terms of R_n , are of the form $m=2^n+k$, $k<2^n$ so one can write that:

$$m^m\equiv m^{2^n}.m^k\equiv m^k\equiv k^k(1+2^n)\pmod{2^{n+2}}$$

Implies that $R_n\equiv(1+2^n)\left(1+3^3+5^5+\cdots.+(2^n-1)^{2^n-1}\right)\equiv(1+2^n)S_n\pmod{2^{n+2}}$

Thus $S_{n+1}\equiv 2S_n\ (1+2^{n-1})\pmod{2^{n+2}}$, now we take an induction on n by induction hypothesis we must have $S_n=2^{n+1}k+2^n$ for some integer k , thus :

$$S_{n+1}\equiv(2^{n+2}k+2^{n+1})(1+2^{n-1})\equiv 2^{n+1}\pmod{2^{n+2}}$$

Ensure that , our proof is complete.

Problem-3. Prove that $\sum_{k=0}^{\left\lfloor\frac{n}{p}\right\rfloor}(-1)^k\binom{n}{kp}$ is divisible by $p^{\left\lfloor\frac{n-1}{p-1}\right\rfloor}$, where p is prime and n is an integer , .

GMA-2013

Solution: assume that $S_n=\sum_{k=0}^{\left\lfloor\frac{n}{p}\right\rfloor}(-1)^k\binom{n}{kp}$, now we first prove preceding lemma:

Lemma: for all $\geq p$: $S_n-\binom{p}{1}S_{n-1}+\binom{p}{2}S_{n-2}+\cdots+\binom{p}{p-1}S_{n-p+1}=0$.

Proof: let ω be primitive $p-\mathrm{th}$ root of unity , we know that $\sum_{i=0}^{p-1}\omega^i=\begin{cases}0 & p\nmid i\\ p & p|i\end{cases}$, thus one can see that $S_n=\frac{1}{p}\sum_{i=0}^{p-1}(1-\omega^i)^n$,since ω^i , $i=1,2,..,p-1$ are roots of the polynomial $\frac{x^p-1}{x-1}$, then $1-\omega^i$ are roots of the polynomial $\frac{1-(1-x)^p}{x}=x^{p-1}-\binom{p}{1}x^{p-2}+\cdots+\binom{p}{p-1}$, by setting $x=1-\omega^i$, $i=0,\dots p-1$ and adding up , for all $n\geq p$ one can find that:

$$S_n-\binom{p}{1}S_{n-1}+\binom{p}{2}S_{n-2}+\cdots+\binom{p}{p-1}S_{n-p+1}=0$$

Our proof is complete.

Back to the problem , we prove the statement by use of the induction on n , it is clear that for $n=1,2,\dots,p-1$. the statement holds , if $n\geq p$ then for all $1\leq j\leq p-1$, $p|\binom{p}{j}$, by induction hypothesis , we now that for all $1\leq j\leq p-1$, $p^{\left\lfloor\frac{n-j-1}{p-1}\right\rfloor}$ divides S_{n-j} , now we know that for all $1\leq j\leq p-1$ $\left\lfloor\frac{n-j-1}{p-1}\right\rfloor+1=\left\lfloor\frac{n+p-j-2}{p-1}\right\rfloor\geq$

$\left\lfloor\frac{n-1}{p-1}\right\rfloor$. Thus all the terms of the sum $\binom{p}{1}S_{n-1} - \binom{p}{2}S_{n-2} + \cdots - \binom{p}{p-1}S_{n-p+1}$ is divisible by $p^{\left\lfloor\frac{n-j}{p-1}\right\rfloor}$ and then that is true for S_n.

Induction and sequences

Problem-1. We know about the sequence $\{x_n\}$ such that $x_1 = \frac{2}{3}$, $x_{n+1} = \frac{3x_n+2}{3-2x_n}$ Is this sequence is eventually periodic? **Bulgarian TST-2011**

Solution: Assume the answer is yes, then there are N, T such that for all $i \geq N$ we have $x_{i+T} = x_i$, we can assume that $N = 1$, indeed since $x_n = \frac{3x_{n+1}-2}{3+2x_{n+1}}$, by $x_{n+1+T} = x_{n+1}$ we deduce that $x_{n+T} = x_n$. lets assume $x_n = \frac{p_n}{q_n}$ in its reduced form , then $x_{n+1} = \frac{p_{n+1}}{q_{n+1}} = \frac{3p_n+2q_n}{3q_n-2p_n}$, lets $d = \gcd(3p_n + 2q_n \,.\, 3q_n - 2p_n)$ then $d = 1\ ,13$. lets define two sequence a_n , b_n such that $a_1 = 2$, $b_1 = 3$, and $a_{n+1} = 3a_n + 2b_n$, $b_{n+1} = 3b_n - 2a_n$, since these sequences are periodic modulo 13 , easy calculation shows that , these sequences hasn't ever divisible by 13 , thus we can say that $p_{n+1} = 3p_n + 2q_n$, $q_{n+1} = 3q_n - 2p_n$, now one can prove inductively that $p_{n+1}^2 + q_{n+1}^3 = 13(p_n^2 + q_n^2) = \cdots . = 13^n$, now if $x_{n+1} = \frac{2}{3}$ then $x_n = 0$ thus $13|p_n$ and then $13|q_n$, a contradiction.

Variant:

We know about the sequence $\{a_n\}$, such that a_0 is a positive integer and we have:

$$a_{n+1} = \begin{cases} \frac{a_n-1}{2} & \text{if } a_n \geq 1 \\ \frac{2a_n}{1-a_n} & \text{if } a_n < 1 \end{cases}$$

We know that $a_n \neq 2$, for $n = 1,2,\ldots,2001$ and $a_{2002} = 2$. Find the value of a_0.

Saint Petersburg

Solution-1: let $a_n = \frac{p}{q}$, then $a_{n+1} = \frac{p-q}{2q}$ or $a_{n+1} = \frac{2q}{q-p}$, since $\gcd(p-q\ ,2q) = 1 \text{ or } 2$ thus in each step , the sum of nominator and denominator either $p+q$, $\frac{p+q}{2}$ thus sum of num and den of a_i doesn't increased , since$a_{2002} = 2$, so sum of num and den are equal to 3we can assume that sum of num and den of a_0 is equal to $a_0 + 1$, so $a_0 + 1 = 3.2^k$, so $a_n = 3.2^{k-n} - 1$ since for $n = 1,2,\ldots,2001$, $a_n \neq 2$ we may have $k = 2002$ and then $a_0 = 3.2^{2002} - 1$.

Solution-2: lets define $b_n = \frac{1}{1+a_n}$ so we can rephrase the statement of the problem as :

$$b_{n+1} = \begin{cases} 2b_n & \text{if } 2b_n < 1 \\ 2b_n - 1 & \text{if } 2b_n > 1 \end{cases}$$

We can preclude the case with $b_n = 1$, which is fixed point of the sequence , so by induction we could find that $b_n = \{2^n b_0\}$, so if we set $1 + a_0 = k$ we have $b_n = \left\{\frac{2^n}{k}\right\}$, since $\left\{\frac{2^{2002}}{k}\right\} = \frac{1}{3}$, we must have $\frac{2^{2002}}{k} - \frac{1}{3} = l \in \mathbb{Z}$, so $3.2^{2002} = k(3l+1)$ thus $k = 3.2^m$ for some positive integer $m \leq 2002$ and then $b_n = \left\{\frac{2^n}{k}\right\} = \left\{\frac{2^{n-m}}{3}\right\}$ since $b_m = \frac{1}{3}$ we find that $m = 2002$, so $k = 3.2^{2002}$ and then $a_0 = 3.2^{2002} - 1$.

Problem-3. We know about sequences $\{x_n\}, \{y_n\}$, such that $x_1 = \frac{1}{10}$, $y_1 = \frac{1}{8}$, $x_{n+1} = x_n + x_n^2$, $y_{n+1} = y_n + y_n^2$, prove that for all n, m . x_n , y_m couldn't be equal. **Saint Petersburg**

First Solution: it is obvious that terms of both sequences are rational numbers assume that $x_n = \frac{p}{q}$ where $\gcd(p, q) = 1$, whence $x_{n+1} = \frac{pq + p^2}{q^2}$ it is obvious that $\gcd(pq + p^2, q^2) = 1$, thus at each step the denominator of x_n is being square and analogously the denominator of y_n , thus denominator of x_n in its reduced form must be power of 10 and denominator of y_n in its reduced form must be power of 8 , thus they couldn't being equal.

Second Solution: since $x_1 = 0.1$, $y_1 = 0.125$ we prove inductively that for all n , x_n ends with 1 and y_n ends with 5 , lets assume the statement being true for all $k \leq n$ thus $x_n = \frac{x}{10^N}$, where x ends with 1 so $x_{n+1} = \frac{10^N x + x^2}{10^{2N}}$, since $10^N x$ ends with zero and x^2 ends with unit , $10^N x + x^2$ ends with unit and our hypothesis is proven, thus for all n , x_n ends with unit , by the same argument we find that y_n ends with 5 , thus they couldn't being equal for all n, m.

Third Solution: since $x_2 = 0.11$, $x_3 = 0.1221$, $x_4 = 0.1221 + 0.1221^2 > 0.13$ so $x_3 < 0.125 = y_1 < x_4$ we prove by induction that : $x_{n+2} < y_n < x_{n+3}$, assume it holds true for all $k \leq n$, then ${x_{n+2}}^2 < {y_n}^2 < {x_{n+3}}^2$ adding the obtained inequality by the original inequality we find that : $x_{n+3} < y_{n+1} < x_{n+4}$. and our claim is proved. Now we receive an infinite sequence $x_3 < y_1 < x_4 < y_2 < x_5 < y_3 < \cdots$ thus no terms could be equal at all.

Induction and Polynomials

Problem-1. Polynomials $F(x), G(x)$ has real coefficients and we know that the points

$$\big(F(1), G(1)\big), \big(F(2), G(2)\big), \ldots, \big(F(2011), G(2011)\big)$$

Are vertex of a 2011-gon in plane , prove that $\deg F$ or $\deg G \geq 2010$.

Solution: assume that the center of 2011-gon is located in $(0,0)$. Thus one can say that :
$F(k) = \mathrm{Cos}(\alpha + \frac{2k\pi}{2011})$, $G(k) = \mathrm{Sin}\left(\alpha + \frac{2k\pi}{2011}\right)$ $(*)$ for $1 \le k \le 2011$. Now by induction on , we show that if for $k = 1,2,\dots,n$ the $(*)$ relations holds $(n \le 2011)$then at least one of F, G hasn't degree less than $n-1$. Now take $n = 2011$ and we are done, for $n = 1$ the statement is true, now assume the statement is true for all ≤ 2010 , let the polynomials F, G satisfies $(*)$ for $k = 1,2 \dots.., n+1$. define $F^*(x) = F(x+1) - F(x)$, $G^*(x) = G(x+1) - G(x)$, now for all $k = 1,2,\dots,n$ we have

$$F^*(k) = -2\sin\frac{\pi}{2011}.\sin(\alpha + \frac{(2k+1)\pi}{2011}),$$

$$G^*(k) = 2\sin\frac{\pi}{2011}.\cos(\alpha + \frac{(2k+1)\pi}{2011})$$

We can divide F^*, G^* by$2\sin\frac{\pi}{2011}$, thus they satisfies the problem criteria , We know that by induction hypothesis at least one of $\deg F^*$ or $\deg G^*(k) \ge n-1$ since $-1 + \deg F = \deg F^*$, $-1 + \deg G = \deg G^*$ we have $\deg F$ or $\deg G \ge n$.

Problem-2.Prove there exist a quadratic polynomial $f(x)$, such that $f(f(x))$ has 4 non-positive real roots , and $\underbrace{f(f(\dots.(x)\dots))}_{n}$ has 2^n real roots. **Bulgarian Olympiads**

Solution: Without less of generality assume that $f(x)$ has positive leading coefficient, (since change the sign of leading coefficient doesn't changes locations of roots) it is obvious that $f(x)$ must have two real roots , say $x_1 < x_2$, if $x_2 > 0$, then there exist a positive real number s such that $f(s) = x_2$, thus $f\big(f(s)\big) = f(x_2) = 0$ and $f(f(x))$ find positive real root , a contradiction , thus we may assume that $x_1 < x_2 \le 0$. since the roots of $f(f(x))$ are indeed , roots of the equations: $f(x) = x_1$, $f(x) = x_2$ and both of the equations must have 2 real roots, lets show the minimal value of the f , by m. So we find that $m < x_1 < x_2 \le 0$. And then roots of $f(f(x))$ lies on the interval (x_1, x_2) Now we complete our proof by induction ,we prove that for all $n > 1$, the polynomial f_n has 2^n real roots lies on the interval (x_1, x_2). For $n = 2$ it is clear , assume $y_1, \dots, y_{2^n}$ being roots of $f_n = \underbrace{f(f(\dots.(x)\dots))}_{n}$ lies on the interval (x_1, x_2), we find that roots of f_{n+1} are roots of $f(x) = y_k$ where $1 \le k \le 2^n$. since $m < x_1$ all the equations of the form $f(x) = y_k$ has two real roots , lies in the interval (x_1, x_2) . whence the polynomial f_{n+1} , has 2^{n+1} real toots lies on the interval (x_1, x_2). We are done.

Problem-3. We know about the polynomial $P(x,y)$ with real coefficients such that , there exist a function f , such that

$$P(x,y) = f(x+y) - f(x) - f(y)$$

Prove that , f can take polynomial values infinitely many times. **Saint-Petersburg Olympiads**

Solution: we define polynomial $f_0(x)$ such that $(x, 1) = f_0(x+1) - f_0(x) - f_0(1)$ and $f_0(1) = f(1)$ it is obvious that one can find all the coefficients of $f_0(x)$ uniquely implies that $f_0(x)$ determines uniquely , now one can find that $f_0(x+1) - f_0(x) = f(x+1) - f(x)$ thus we can inductively prove that $f(n) = f_0(n)$ for all positive integers n , thus for all positive integers x, y we have $P(x, y) = f_0(x+y) - f_0(x) - f_0(y)$ and we are done.

On the degree of Polynomials

Problem-1. Prove that there exist positive real numbers $a_0 , a_1, \dots , a_n$ such that the polynomial $\pm a_n x^n \pm a_{n-1}x^{n-1} \pm \cdots . \pm a_0$ for all choose of sign has n real roots. **IMC-2014**

Solution: we prove the problem statement , by use of induction on n , by induction hypothesis , the polynomial $\overline{P(x)} = \pm a_n x^{n+1} \pm a_{n-1}x^{n-1} \pm \cdots . \pm a_0 x$, where $a_0 \neq 0$, has $n+1$ distinct real roots , thus it hasn't any double root ,thus any of its local extreme wasn't its roots , so for all polynomial $\overline{P(x)}$, for all of its n extremes $s_1 , \dots , s_n$ there are $\varepsilon > 0$, such that $\left|\overline{P(s_i)}\right| > \varepsilon$, so we prove all of the polynomial $P(x) = \pm a_n x^{n+1} \pm a_{n-1}x^{n-1} \pm \cdots . \pm a_0 x \pm \varepsilon$, has exactly $n+1$ real roots , since $\overline{P(x)}$ has local extremum $s_1 < \cdots < s_n$, such that $P(s_i), P(s_{i+1})$ has different signs thus it has roots in any interval of (s_i , s_{i+1}) adding up $(-\infty , s_1), (s_n , +\infty)$ .so by induction , our proof is complete.

Variant:

Is there exist a sequence of nonzero real numbers $a_0 , a_1, \dots , a_n, \dots$ such that for all n the polynomial $P_n(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots . + a_0$ has n distinct real roots ? **Putnam-1990**

Solution: besides previous solution , we can construct an infinite sequence $a_n = (-1)^n 10^{-n^2}$, thus $(-1)^k 10^{-k^2} P(10^{2k}) = \sum_{i=0}^{n} (-1)^{i-k} 10^{-(i-k)^2} = \sum_{j=-k}^{j=k} (-1)^j 10^{-j^2} > 1 - 2\sum_{j=1}^{\infty} 10^{-j^2} > 0$

Henceforth , $P(1), \dots , P(10^{2n})$ changes theirs signs , alternatively....

Problem-2. We know about function $f(x)$ such that for all integer x , we have $f(x) \in \mathbb{Z}$, for all prime p there exist a polynomial $Q_p(x)$ of degree less than or equal 2013 , with integer coefficients such that $f(n) - Q_p(n)$ is divisible by p , prove that there are polynomial $g(x)$ with real coefficients such that for all positive integers we have $f(n) = g(n)$. **Moskow-2013**

First Solution: we prove this problem , by induction on degree of polynomial $Q_p(x)$ such that for all polynomial $Q_p(x)$ of degree less than or equal to k , $f(x)$ takes polynomial values in positive integer points. For $= 0$, $Q_p(x)$ must be constant and $f(n) - Q_p(n) = f(n) - c$ is divisible by p thus for all $m, n \in \mathbb{N}$ $f(m) - f(n)$ is divisible by p , so take p large enough , we conceive that $f(x)$ must be constant integer. Now we prove two following lemma:

Lemma-1: let $h(x)$ be a polynomial of degree , then the polynomial $\Delta h = h(x+1) - h(x)$ is of degree $d-1$.

Proof: the prof is indeed obvious.

Lemma-2: if for all integer , the function $\Delta h(x)$ takes a polynomial value of degree less than or equal to $d-1$, then for all integer x , we have $h(x) = h_1(x)$ for some polynomial h_1 of degree less than or equal to d.

Proof: we prove this lemma by induction on , for $d = 1$, we can say that $h(x) = h(0) + cx$ for all integer x . now assume $\Delta h(x) = ax^d + \cdots$ define $h_0(x) = h(x) - \frac{a}{d+1} x(x-1) \ldots . (x-d)$ then

$$\Delta h_0 = \Delta h(x) - \ ax(x-1) \ldots . (x-d+2)$$

Then $\deg \Delta h_0 \ \leq d-1$, so by induction hypothesis , h_0 is equal to polynomial of degree less than or equal to d since $\deg h_0(x) - h(x) \leq d+1$, whence $h(x)$ in integral points take the value of a polynomial of degree less than or equal to $d+1$, ensure that our proof is complete.

Take back to our problem , since the function Δf satisfies the statement of the problem (indeed $\Delta f(x) - \Delta Q_p(x)$) is divisible by p in positive integer points , note that $\deg \Delta Q_p(x) \leq k-1$. so by induction hypothesis $\Delta f(x)$ take polynomial values . thus by lemma-2, the polynomial $f(x)$ takes polynomial value of degree less than or equal one unit added up to whatever polynomial $\Delta f(x)$ takes its values on integral points.

Second Solution: by use of lagrange interpolation formula one can find that we can find the polynomial $f_0(x)$, by the values of $f(1), \ldots . , f(2014)$ with rational coefficients of degree less than or equal to 2013 , now if we take $c = 2013!^2$, then one can easily find that $cf_0(x)$ has integer coefficients , take a prime $p > c$, then $\deg cQ_p(x) - cf_0(x) \leq 2013$, and has 2014 incongruent roots modulo p , thus polynomial $cQ_p(x) - cf_0(x)$ is zero polynomial modulo all large prime p note that:

$$c\left(f(x) - Q_p(x)\right) + c\left(Q_p(x) - f_0(x)\right) = c(f(x) - f_0(x))$$

Is divisible by sufficiently large , and thus one can find that for all integer x , we must have: $f(x) = f_0(x)$. (fix x and take p , sufficiently large).

Problem-3. Prove that If a rational function that is not a polynomial assumes rational values at all positive integral points, then it is the quotient of two relatively prime polynomials with integral coefficients.

Solution: Let the function in question be $R(x) = \frac{P(x)}{Q(x)}$ where $P(x), Q(x)$ are relatively prime polynomials, r the sum of the degrees of $P(x)$, $Q(x)$. For $r = O$ the theorem is obvious. Consider if necessary $\frac{1}{R(x)}$ instead of $R(x)$ and assume that the degree of $P(x)$ is not less than that of $Q(x)$ and furthermore that a is a positive integer such that $Q(a) \neq 0$ Then $\frac{P(a)}{Q(a)}$ is rational and the rational function $\frac{1}{x-a}\left(R(x) - \frac{P(a)}{Q(a)}\right) = \frac{P_1(x)}{Q(x)}$ is a rational function whose value is rational for integral $x, x > a$, and the degree of $P_1(x) = \frac{Q(x)Q(a) - Q(x)P(a)}{Q(a)(x-a)}$ is less than that of P(x); thus the sum of the degrees of $P_1(x), Q(x)$ is less than that of $P(x)$ and $Q(x)$. so by induction we are done.

Problem-4. Prove that If a rational function has integral values for infinitely many integral values of the variable, then it is a polynomial.

Solution: The function $f(x) = \frac{P(x)}{Q(x)}$ is considered , where $P(x)$ and $Q(x)$ are polynomials with integral coefficients. We can find an integer q such that $qf(x) = G(x) + r(x)$, where $G(x)$ denotes a polynomial with integral coefficients and $r(x)$ a rational function whose numerator is of lower degree than its denominator. The value of $r(x)$ is integral for infinitely many integral values of x. Since $\lim_{x\to\infty} r(x) = 0$. from a sufficiently large points we must have $|r(x)| < 1$, thus $r(x) = 0$ and hence is zero for all $x$for a rational function . so we are done.

Miscellaneous

Problem-1. Let $P(x) = a_d x^d + \cdots . + a_0$ prove that the polynomial is divisible by $(x-1)^m$ if and only if for all $s = 0,1,..,m-1$. we have $a_n(n+1)^s + a_{n-1}n^s + \cdots + a_1 2^s + a_0 = 0$

Solution: let $(x) = (x-1)^m Q(x)$, where $Q(x) = \sum_{i=0}^{n-m} q_i x^{n-m-i}$ since $(x-1)^m = \sum_{j=0}^{m} \binom{m}{j} x^{m-j}$, then:

$P(x) = \left(\sum_{i=0}^{n-m} q_i x^{n-m-i}\right)\left(\sum_{j=0}^{m} \binom{m}{j} x^{m-j}\right) = \sum_{i=0}^{n-m} q_i \sum_{j=0}^{m} (-1)^j \binom{m}{j} x^{n-i-j}$. The problem statement reduces to : $\sum_{i=0}^{n-m} q_i \left(\sum_{j=0}^{m} (-1)^j \binom{m}{j} (n-i-j+1)^s\right) = 0$. the term , $\sum_{j=0}^{m} (-1)^j \binom{m}{j} (n-i-j+1)^s)$ is equal to $m - \text{th}$, difference of the polynomial x^s , which is identically zero.

Proof of Sufficiency: we prove this by induction on , the case $m = 1$ is clear , assume the statement is true for $m = k$ and divide $P(x)$ by $(x-1)^{k+1}$ now we have $P(x) = (x-1)^{k+1} q(x) + r(x)$ where $\deg r(x) \leq k$. by induction hypothesis , we know that $P(x)$ is divisible by $(x-1)^k$ so $r(x) = \mathrm{a}(x-1)^k$ the necessity condition depicts that $(x-1)^{k+1} q(x)$ satisfies the conditions for $s = 0,1, \ldots, k$. now we only need to check the condition for $r(x)$, for $s = k$, we have $r(x) = a \sum_{j=0}^{k} (-1)^j \binom{k}{j} x^{k-j}$ thus we have :

$$a(\,(k+1)^k - k.k^k + \cdots.) = 0$$

The expression in the bracket is equal to ! . thus $a = 0$.

Comment: Instead of $1,2, \dots, n+1$we can use the $n+1$ consecutive terms of Arithmetic progression.

Problem-2. let P(x)has n-distinct real root larger 1 and smaller 0,prove that:

1.P'(x) has a root greater than $1 - \frac{1}{n}$.

2.Absolute value of difference between maximum and minimum root of P'(x)is greater than $\sqrt{1 - \frac{2}{n}}$.

Saint Petersburg

Solution:

Induction and functional equations

Problem-1. Find all function $f: \mathbb{N} \to \mathbb{N}$ such that

$$2f(mn) \geq f(m^2+n^2) - f^2(m) - f^2(n) \geq 2f(m)f(n)$$

Silk Road Math Competitions

Solution: define $= \{k|\, f(kn) = k^2 f(n)\}$, $A_1 = \{k|\, f(k) = k^2\}$, from the above inequalities we find that
1. $f(mn) \geq f(m).f(n)$
2. $f(n^2+m^2) \geq (f(m)+f(n))^2$
3. $f^2(m) + f^2(n) + 2f(mn) \geq f(n^2+m^2)$

Now we prove following lemmas:

Lemma-1: if $f(mn) = f(m).f(n)$ then $f(n^2+m^2) = (f(m)+f(n))^2$.

Proof: by three inequalities , it is clear.

Now take $= n = 1$, then $f(1) = 1$, set $m = 1$ in the above lemma we receive $f(n^2+1) = (1+f(n))^2$ (4) , set $n = 1$ we find that $f(2) = 4$, take $m = n$ in the inequalities $1,3$ we find that : $f^2(n) \geq f(n^2)$, $f(n^2) \geq f^2(n)$. thus $(n^2) = f^2(n)$. we prove following lemma:

Lemm-2: if for all , $f(kn^2) = k^2 f^2(n) = k^2 f(n^2)$ then . $k \in A$.

Proof: $(k) = f(k.1^2) = k^2 f^2(1) = k^2$, and $k^2 f^2(n) = f(kn^2) \geq f(kn). f(n)$

so $f(kn) \leq k^2 f(n)$, on the other hand $f(kn) \geq f(k). f(n) = k^2 f(n)$ so $f(kn) = k^2 f(n)$,, whence $k \in A$.

We know $f(n^2) = f^2(n)$ so $1 \in A$, then by lemme-1 , for $m = n$ we have $f(2n^2) = 4f(n^2)$ so $2 \in A$, conclude that $f(2n) = 4f(n)$. Now set $n = p_1 \dots . p_t$ where p_i are primes , we know that

$$f(n) \geq f(p_1) \dots . f(p_t)$$

We prove by induction that $(n) \geq n^2$, it is clear for $n = 1,2$. let $n = mk$ where $1 < m, k < n$ by induction hypothesis $f(n) \geq f(m). f(k) \geq m^2 k^2 = n^2$. if $n \equiv 1 \pmod 4$ there are integers x, y such that $n = x^2 + y^2$ and $x, y < n$ .thus by induction hypothesis :

$$f(n) = f(x^2 + y^2) \geq (\, f(x) + f(y))^2 \geq (x^2 + y^2)^2 = n^2$$

If $n \equiv 3 \pmod 4$ then $n^2 + 1 = 2p_1 \dots p_t$ where $p_i \equiv 1 \pmod 4$ which wasn't necessarily distinct , so there are positive integers x_i, y_i such that $x_i^2 + y_i^2 = p_i \leq \frac{n^2+1}{2} < n^2$ so $x_i, y_i < n$, then by induction hypothesis : $f(p_i) = f(x_i^2 + y_i^2) \geq (f(x_i) + f(y_i))^2 \geq (x_i^2 + y_i^2)^2 = p_i^2$ .thus :

$$f(n^2 + 1) \geq f(2). f(p_1) \dots . f(p_t) \geq 4p_1^2 \dots p_t^2 = (n^2 + 1)^2$$

Since $f(n) = \sqrt{f(n^2 + 1)} - 1$ we have $f(n) \geq n^2 + 1 - 1 = n^2$. We are done.

Now we complete our proof with two different methods:

Method-1: we use induction to prove $f(n) = n^2$ assume it holds true for all $1 \leq i \leq m$, now one can see that $f\big(2(m^2 + 1)\big) = 4f(m^2 + 1) = 4(m^2 + 1)^2$ now set $n = m - 1 , m = m + 1$ in the inequality (2) we find that

$$4(m^2 + 1)^2 = f(2\big(2(m^2 + 1)\big) = f(\,(m - 1)^2 + (m + 1)^2) \geq (f(m + 1) + f(m - 1))^2 = (f(m + 1) + (m - 1)^2)^2$$

Thus we can find $f(m + 1) \leq (m + 1)^2$ but we proved $(m + 1) \geq (m + 1)^2$. so the equality occurred . and our inductive proof is complete.

Method-2: we continue our proof , by proving thi lemma :

Lemma-3: if $a, \in A$ then so is $ab, a^2 + b^2$.

Proof: since $f(an) = a^2 f(n), f(bn) = b^2 f(n)$ we find that $(abn) = a^2 f(bn) = (ab)^2 f(n)$. we know that $x = an , y = bn$ satisfies the equality $f(xy) = f(x). f(y)$ hence by lemma-1 we find that

$$f\left((a^2+b^2)n^2\right) = f(x^2+y^2) = (f(x)+f(y))^2 = (a^2+b^2)^2 f^2(n)$$

Then our proof is complete.

Lemma-4: if $k \in A$ and $d|k$ then $\in A$.

Proof: $k^2 f(n) = f(kn) \geq f(dn).f\left(\frac{k}{d}\right) \geq (\frac{k}{d})^2 f(dn)$ therefore $d^2 f(n) \geq f(dn) \geq f(d).f(n) \geq d^2 f(n)$ .thus $f(dn) = d^2 f(n)$ and it means that $d \in A$.

We construct set of primes $p_1 = 2$, $p_{i+1}|(p_1 \dots p_i)^2 + 1$ by induction on i and use of lemma- 3 and lemma -4, we can see that $p_i \in A$.

Lemma-5: let $r = a^2 + b^2 \in A_1$ then $a, b \in A_1$

Proof: since $r \in A_1$ then $f(r) = r^2 = f(a^2+b^2) \geq (f(a)+f(b))^2 \geq (a^2+b^2)^2 = r^2$ so the equality occurred and then $f(a) = a^2$, $f(b) = b^2$.

Lemma-6: if $r \in A_1$ and $d|r$ then $d \in A_1$

Proof: write $r = kd$, then $d^2 \leq f(d) \leq \frac{f(r)}{f(k)} \leq \frac{r^2}{k^2} = d^2$ so $(d) = d^2$. shows that $d \in A_1$.

Now since there are infinitely many $p_i \equiv 1 \pmod 4$ in the set A , we can find that $f(p_i) = p_i^2$ we know that for each of which there are integers x_i, y_i such that $x_i^2 + y_i^2 = p_i$. take arbitrary positive integer n and consider $\gcd(n, y_i)$, whence there is a certain number $d$which occurred infinitely many times , define the set

$$B = \{i | \gcd(n, y_i) = d\}$$

So $y_i = dz_i$ and $\gcd\left(z_i, \frac{n}{d}\right) = 1$ whence there are $i < j \in B$ such that $\frac{x_i}{z_i} \equiv \frac{x_j}{z_j} \pmod{\frac{n}{d}}$ set $s = |x_i y_j - x_j y_i| = d|x_i z_j - x_j z_i| \equiv 0 \pmod n$ now set $= x_i x_j + y_i y_j$, so $p_i p_j = s^2 + t^2$ we know that $p_i p_j \in A$ and $f(p_i p_j) = (p_i p_j)^2$ so $p_i p_j \in A_1$ so $s, t \in A_1$. we now that $n|s$ whence by lemma-6 , $n \in A_1$ then $f(n) = n^2$.

Finally Our long-run proof is complete!

Problem-2. Prove that for all positive integers $a, b > 1$, there are a function $f: \mathbb{N} \to \mathbb{N}$ such that :

$$f\left(af(n)\right) = bn$$

Brazilian Olympiad

Solution: define two increasing sequence $\{S_n\}_{n=1}^{\infty} = \mathbb{N} - \{a, 2a, 3a, \dots.\}$, $\{R_n\}_{n=1}^{\infty} = \mathbb{N} - \{b, 2b, \dots\}$ in turn , of non multiples a, b . one can easily find that , if for example $R_l = R_k$, then $l = k$. we prove that the following function satisfies the problem conditions:

$$f(n) = \begin{cases} R_k & a \nmid n \,, S_k = n \\ bS_l & n = ak \,, b \nmid k \,, R_l = k \\ abf(j) & n = abj \end{cases}$$

Now we prove that the above function is indeed a solution , we divide our proof in 3 cases:

Case-1: $a \nmid n$. We can write $S_k = n$. Then $f\big(af(n)\big) = f(aR_k) = bS_l = bn$ (since $\{S_n\}_{n=1}^{\infty}$ is an increasing sequence , then $k = l$)

Case-2: $n = ak$, $b \nmid k$. Then $f\big(af(n)\big) = f\big(af(ak)\big) = f(abS_l), k = R_l$ now $f(abS_l) = abf(S_l) = abR_k$, $(S_k = S_l)$ thus $f(abS_l) = abR_l = abk = nb$

Case-3: -$n = abj$, then $f(af(n)) = f\big(af(abj)\big) = f(ab\big(af(j)\big) = abf(af(j))$ now if $ab \nmid j$ then by above cases :$f\big(af(j)\big) = bj$ and we are done . if $ab|j$ set $j = abi$ and then $i < j$ now we use induction on j and by use of induction hypothesis on i we can find

$$f\big(af(j)\big) = f\big(af(abi)\big) = f(ab\big(af(i)\big) = abf\big(af(i)\big) = ab.bi = bj$$

So , if $= abj$ then $f\Big(f\big(af(n)\big)\Big) = ab.bj = b.n$

Problem-3. Find all function : $\mathbb{R}^+ \to \mathbb{R}$, such that $f(x) + f(y) \leq \frac{1}{2} f(x+y)$, $\frac{f(x)}{x} + \frac{f(y)}{y} \geq \frac{f(x+y)}{x+y}$

Japanese Olympiads-2007

Solution: set $= y = t$, then $4f(t) \leq f(2t)$ and $2\frac{f(t)}{t} \geq \frac{f(2t)}{2t}$ then $f(2t) = 4f(t)$ by induction it could be find that $f(2^m t) = 2^{2m} f(t)$. for all positive integer m. Define $g(x) = \frac{f(x)}{x}$ we now that for $= 2^m$, $m \in \mathbb{N}$, we have $g(nt) = ng(t)$ since $g(x) + g(y) \geq g(x+y)$, so inductively , $g(nt) \leq ng(t)$. now take $2^m > n$, then $g(2^m t) \leq g(nt) + g\big((2^m - n)t\big) \leq ng(t) + (2^m - n)g(t) = 2^m g(t) = g(2^m t)$ see the equality occurs , thus we can see that $g(2^m t) = g(nt) + g\big((2^m - n)t\big)$ thus for all integer n , we have $g(nt) = ng(t)$. We know that $f(2t) = 4tg(t), f(3t) = 9tg(t)$ take $x = 2t$, $y = t$ in the first inequality then $f(2t) + f(t) \leq \frac{f(3t)}{2}$ thus $5tg(t) \leq \frac{9}{2} tg(t)$, then $g(t) \leq 0$ for all positive t.

Then take $0 < x \le y$ so $g(x) \ge g(x) + g(y-x) \ge g(y)$ so $g(x)$ is a decreasing function . take $(1) = a \le 0$. then for all positive integer n m we find that $g(n) = a.n$, if $\gcd(p,q) = 1$. then $q.g\left(\frac{p}{q}\right) = g(p) = a.p$ so $g\left(\frac{p}{q}\right) = \frac{a.p}{q}$. we also know that for all $t > \frac{p}{q}$, we have $g(t) < \frac{a.p}{q}$. if there are $t \in \mathbb{R} - \mathbb{Q}$ such that $g(t) < at,$ there exist a rational r , such that $g(t) < r < at$, since g is decreasing we have

$$ar = g(r) > g(at) = ag(t)$$

Since a isnt positive we must have $< g(t)$, a contradiction , analogously for the case of existence of$t$ forwhich $g(t) > at$. Thus $g(t) = at$. For all real . thus $f(t) = at^2$ clearly satisfies the conditions of the problem.

On sum of num and den

Problem-2. Find all function $f\colon \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f\left(\frac{x}{x+1}\right) = \frac{f(x)}{x+1}$, $f(x) = x^3 f(\frac{1}{x})$

Turkish Olympiads

Solution: we prove that the only functions satisfies the problem conditions are $f_a(x) = a.\frac{n^2}{d}$ where d, n are relatively prime positive integers , it is easy to check , function $f_a(x)$ satisfies . now we made an induction on $\max\ (n,d)$ (or $n+d$ respectively) for $n = d = 1$, we have $f(1) = a = f_a(1)$. Now we prove inductively that for all ≥ 2 , we must have $f\left(\frac{n}{s}\right) = f_a\left(\frac{n}{s}\right)$ where $\gcd(n,s) = 1$. assume the statement was true for $\mathrm{k} = \max\ (n,d) < k$, now if $\gcd(n,d) = 1$, $\max\ (n,d) = k$ we know that $n \ne d$, so we have two case , First. If $< d$, then $\gcd(n, d-n) = 1$ so $f\left(\frac{n}{d}\right) = f\left(\frac{\frac{n}{d-n}}{1+\frac{n}{d-n}}\right) = \frac{f(\frac{n}{d-n})}{\frac{d}{d-n}} = a.\frac{n^2}{d}$. Second. If $d < n$ then $\left(\frac{n}{d}\right) = f\left(\frac{1}{\frac{d}{n}}\right) = \frac{f(\frac{d}{n})}{\frac{d^3}{n^3}} = a.\frac{n^2}{d}$. now our proof is complete.

Problem-3. When we have number x in hand , two operation $x \to \frac{1+x}{x}$, $x \to \frac{1-x}{x}$, is permitted , Is it true that each of the nonzero rational number can be obtained with a finite number of such operations ? **Moskow-Olympiads-2007**

Solution: lets $f(x) = \frac{1+x}{x}$, $g(x) = \frac{1-x}{x}$ one can see that $f(g(f(x)) = -x$, $f(g\left(f\left(g(f(x))\right)\right)) = \frac{1}{1+x}$ so $f(g\left(f((f(g(f(x))))\right) = x\ (x \ne -1)$, $f(g\left(f\left(g\left(f(g(x))\right)\right)\right) = x$, $(x \ne 1)$

Now number 2 could be obtain from -2 and vice versa , since $f(-2)=\frac{1}{2}$, $g\left(\frac{1}{2}\right)=1$ we can receive number 1 from 2 , besides , since $g(-1)=-2$, and -2 can get2 from -1 . since

$$g(2)=\frac{-1}{2}\ ,f\left(\frac{-1}{2}\right)=-1$$

we can get -1 from -2 . the cycle is complete(i.e. $2 \leftrightarrow 1$, $\underbrace{-2 \rightarrow -1 \rightarrow 2}_{2\leftrightarrow -2}$)

now by induction on sum of num and den of the positive irreducible fraction $\frac{m}{n}$, we construct all positive rational number then by the equality $f(g(f(x))=-x$ we construct all negative ones. For $m+n=2$ we are done , assume the statement holds true for all $m+n<k$. so if , $m>n$ then $\frac{m}{n}=1+\frac{1}{\frac{n}{m-n}}$ since $n+m-n=m<k$ we can construct $\frac{n}{m-n}$ and so forth $\frac{m}{n}$ (by f). if $m<n$ then $\frac{m}{n}=\frac{1}{1+\frac{n-m}{m}}$ since $n-m+m=n<k$. We can construct $\frac{n-m}{m}$ and so forth$\frac{m}{n}$ (by $f(g\left(f\left(g(f(x))\right)\right)))$. and we are done.

Induction and Number Theory

Problem-1.Prove that for all positive integer n there exist an integer number such that decimal expansion of its square starts with n ones and ends with a block of length n with combinations of ones and two s.

Moskow-2014

Solution: we prove this by induction on , such that for all $n$there exist an integer m_n whose decimal expansions ends with units and decimal expansion of m_n^2ends with a block of length n with combinations of ones and two s.

for $n=1$ it is clear , assume the statement of the problem holds for $n=k$, and the desired number was m_k , lets define $p_a=m_k+a.10^k$ for some $a\in\{1,2\ldots.,9\}$ ends with unit . thus $p_a^2=m_k^2+2a.m_k.10^k+a^2.10^{2k}$, the number m_k^2 ends with k ones and twos , the number $2a.m_k.10^k$ ends with k zeros and $a^2.10^{2k}$ ends with $2k$ zeros , name the $k+1-$ th digits of m_k^2 by b and $k+1-$th digits of $2a.m_k.10^k$ is $2a(\text{mod } 10)$(since m_k ends with unit) thus $k+1-$ th digits of p_a^2 is $b+2a(\text{mod } 10)$ if b odd take a such that last digits of $b+2a(\text{mod } 10)$ being 1 , if b being even , take a such that last digits of $b+2a(\text{mod } 10)$ is 1,0 . thus we can take $p_a=m_{k+1}$.

Now set $c_n=\underbrace{11\ldots.1}_{n}.10^{4n}$ and $d_n=c_n+10^{4n}$, now:

$$\sqrt{d_n} - \sqrt{c_n} = \frac{d_n - c_n}{\sqrt{d_n}+\sqrt{c_n}} = \frac{10^{4n}}{\sqrt{d_n}+\sqrt{c_n}} > \frac{10^{4n}}{2.10^{3n}} > 1$$

Thus there exist an integer in the interval $(\sqrt{c_n}, \sqrt{d_n})$ such that its square starts with n units , show by p_n now construct the number $p_n. 10^k + m_n$ take k large enough (i.e. larger than number of digits of $2p_n m_n$, m_n^2 . now one can see that $(p_n. 10^k + m_n)^2 = p_n^2. 10^{2k} + 2p_n m_n. 10^k + m_n^2$, now the first n digits of this number characterize by p_n^2 (string of units) and the last n digits of that number is characterized by m_n^2(strings of units and twos)

Problem-2. Prove that for all positive integer , there exist, integer k such that $S(k) = n, S(k^2) = n^2, S(k^3) = n^3$.

Moskow-Olympiads-2013.

Solution: we prove more general statement : for any positive integer n there exist a nonnegative integer k consist only of $0,1$ and k^2 consist only of $0,1,2$ in theirs decimal representations such that :

$$S(k) = n, S(k^2) = n^2, S(k^3) = n^3.$$

the statement by induction on n , for $n = 1$ it is obvious , assume the statement holds true for n , now we must prove that there exist an integer m such that $S(m) = n + 1, S(m^2) = (n + 1)^2, S(m^3) = (n + 1)^3$. Lets name the number which satisfies the statement for n by t and assume that t has k digits , now take $m = t + 10^{10k}$, then we cans see that $S(m) = 1 + S(t) = n + 1$ and

$$S(m^2) = S(t^2 + 2t. 10^{10k} + 10^{20k}) = S(t^2) + 2S(t) + 1 = (n + 1)^2$$

Since , $t^2 < 10^{2k}$,$2t. 10^{10k} < 10^{20k}$, numbers$t^2, 2t. 10^{10k}, 10^{20k}$ hasn't any in-common digits which affects theirs addition. And m^2 consists only of $0,1,2$. now we can see that

$$m^3 = t^3 + 3t^2. 10^{10k} + 3t. 10^{20k} + 10^{30k}$$

By the same argument , one can find that $S(m^3) = (n + 1)^3$.

Problem-3. Let a, b, c, m, n being positive integers and $f(x) = ax^2 + bx + c$, prove that there exist n consecutive integers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)$ has at least m distinct prime factors.

Solution: we make an inductive reasoning on m, for $m = 1$ the statement of the problem holds, now choose $\alpha_1, \alpha_2, \dots, \alpha_n$ such that $f(\alpha_1), f(\alpha_2), \dots, f(\alpha_n)$ has at least m distinct prime divisors , now take $A = f^2(\alpha_1). f^2(\alpha_2). \dots . f^2(\alpha_n)$ and $\beta_j = A + \alpha_j$, then $f(\beta_j) = f(\alpha_j). (\underbrace{1 + B. f(\alpha_j)}_{*})$, for some integer B thus the $*$ term is coprime to $f(\alpha_j)$ and then has at least one prime number other than $f(\alpha_j)$. So we are done.

Problem-4. Let $\sigma(m) = \sum_{d|m,0<d<m} m$ prove that there exist a positive integer m such that for all , we have:

$$m < \sigma(m) < \sigma(\sigma(m)) < \cdots . < \sigma^{(t)}(m)$$ **AMM**

Problem-5. Let p being odd prime and $a_1, \dots a_{2p-1}$ being distinct numbers in the interval $[1, p^2]$ which theirs sum is divisible by . prove that there exist positive integers $b_1, \dots b_{2p-1}$ none of which is divisible by p and theirs decimal representations in base p has only 0 , 1 such that $\sum a_i b_i$ being divisible by p^{2012}.

Alexander Ivanov, Bulgarian Olympiads

Problem-6. Prove that 2010 positive integers whose every sum of the integers selected from them is perfect power greater than unit.

Poolish Olympiads

Solution: We prove inductively that for all there is a set S_k with such properties . for arbitrary set T , call a subset X of T good if sum of elememts of this subset be a power number , for $S_1 = \{4\}$, the statement holds , assume there is a set S_k with desired properties , define $U_1 = S_1 \cup \{b\}$ if some subsets of U_1 wasn't good , we start to dis-eviling them , we construct the set U_i by multiplying elements of U_{i-1} by c_i , assume that in the set U_i there are t good subset which theirs sum is $m_1, \dots, m_t$ power of integers , multiply all the elements by $c^{\mathrm{lcm}(m_1,\dots,m_t)}$ now if sum of one non-good subset is c in U_i , now it is $c^{1+\mathrm{lcm}(m_1,\dots,m_t)}$ in U_{i+1} , now the set U_{i+1} is defines as $U_{i+1} = \{a.c^{\mathrm{lcm}(m_1,\dots,m_t)} | a \in U_i\}$ thus in each step the number of good subsets , increased , after finite steps say m ,the set U_m has all subsets good , and then set $S_{k+1} = U_m$. So we are done.

Problem-7. Let C being a positive integer and p_1 being prime number and for all > 1 , p_n is the prime divisor of $p_{n-1} + C$, Which not enlisted in the sequence before , prove that sequence $p_1, p_2, \dots$is bounded. **Tuymada-2004**

Solution: consider , consecutive composite numbers $A = N(C+1)! + 2, N(C+1)! + 3, \dots, N(C+1)! + C + 1$, and take natural number N , sufficiently large such that $A > p_1, C$. we prove by induction that for all natural number n , we must have $p_n < A$. for $n = 1$, it holds true , assume the statement true for natural numbers less than n m but assume contrary that $p_n \geq A$, then $p_{n-1} + C \geq p_n \geq A$. since $p_{n-1} < A$ we find that $p_{n-1} + C$ will fall in the above composite sequence , so must be composite whence :

$$p_n \leq \frac{p_{n-1}+C}{2} < \frac{A+A}{2} = A$$

A contradiction , so for all integer n , we have $p_n < A$.

Problem-8. Prove that there exist positive integers $a_1 < a_2 < \cdots .. < a_{2011}$, such that for all $1 \leq i < j \leq 2011$, such that $\gcd(a_i, a_j) = a_j - a_i$ **Brazilian Olympiads**

Solution: At first we prove that the condition of the problem is equivalent to $a_j - a_i | a_i$ (*), if $\gcd(a_i, a_j) = a_j - a_i$ then $a_j - a_i | a_i$ now if $a_j - a_i | a_i$ then $a_j - a_i | a_j$ so $a_j - a_i | \gcd(a_i, a_j)$ since $\gcd(a_i, a_j) \le a_j - a_i$ we are done. Now we construct these numbers inductively, the case $n = 2$ is obvious, now assume we construct $x_1 < x_2 < \cdots .. < x_{k-1}$ such that $\gcd(x_i, x_j) = x_j - x_i$ now we put k numbers

$$x_0 < x_0 + x_1 < x_0 + x_2 < \cdots . < x_0 + x_{k-1}$$

Where x_0 will be determined later, since the problem condition is equivalent to (*) we must have $x_j - x_i | x_0 + x_i$, $x_i | x_0$ take $x_0 = \operatorname{lcm}(x_1, x_2, \dots\dots, x_{k-1})$ and we prove tha statement for $n = k$.

Problem-9. Let $a, m \ge 2$, $\gcd(a, m) = 1$ and $ord_{\mathrm{m}}^{\mathrm{a}} = k$, t being an odd number and every prime dividing t, also divides m, but $\gcd\left(t, \frac{a^k - 1}{m}\right) = 1$, prove that $ord_{\mathrm{mt}}^{\mathrm{a}} = kt$. **Bulgarian TST**

Solution: we prove this problem by induction on number of prime factors of , with theirs multiplicity . if t being prime, $d = \frac{a^k - 1}{m}$, $t|m$, set $a^k = 1 + md$, thus $a^{kt} = (1 + md)^t \equiv 1 \pmod{mt}$ thus $s = \operatorname{ord}_{mt}^{a} | kt$ we know that $mt | a^s - 1$ thus $m | a^s - 1$ sice $k = \operatorname{ord}_m^a$ then $k|s|kt$ so $s = k$ or kt. if $s = k$ then $1 + md = a^k \equiv 1 \pmod{mt}$ this implies that $t|d$, a contradiction. So $= kt$.

Now assume that t has at least to prime factor(not necessarily distinct) write $t = rt_0$ where r is prime and $t_0 > 1$ since r is prime by the use of base of the induction we find that $ord_{\mathrm{mr}}^{\mathrm{a}} = kr$. at first we prove $\gcd\left(t_0, \frac{a^{kr} - 1}{mr}\right) = 1$, then we will use the induction hypothesis . if prime number r_0 divides t_0 it also divides t so $r_0 | m | mr$, now lets regard : $d_0 = \frac{a^{kr} - 1}{mr} = \frac{a^k - 1}{m} . \frac{a^{k(r-1)} + a^{k(r-2)} + \cdots . + 1}{r} = d . \underbrace{\frac{a^{k(r-1)} + a^{k(r-2)} + \cdots . + 1}{r}}_{c}$ since $a^k \equiv 1 \pmod{m}$ then $a^k \equiv 1 \pmod{r}$ thus c is an integer , if $r \ne r_0$ then $a^k \equiv 1 \pmod{r_0}$ thus $c \equiv 1 \pmod{r_0}$ so $\gcd\left(t_0, \frac{a^{kr} - 1}{mr}\right) = 1$ and we can use induction hypothesis . if $r = r_0$ then one can set $a^k \equiv 1 + br \pmod{r^2}$ for some integer b, thus for all $j = 0, 1, \dots, r - 1$ by use of binomial theorem we can find that $a^{kj} \equiv 1 + jbr \pmod{r^2}$ .so

$$a^{k(r-1)} + a^{k(r-2)} + \cdots . + 1 \equiv r + br(1 + 2 + \cdots + r - 1) \equiv r + \frac{b}{2} r^2 (r - 1) \pmod{r^2}$$

Thus $c \equiv 1 \pmod{r}$ and again $\gcd\left(t_0, \frac{a^{kr} - 1}{mr}\right) = 1$, so our inductive reasoning is complete.

Problem-10. Prove that for all positive integers m, n there exist an integer , such that $2^k - m$ has at least n distinct prime divisors. **Chinese TST**

Solution: we prove the statement by induction on , for $n = 1$ the statement is clear , assume now that there is an integer k_n such that $A_n = 2^{k_n} - m$ has at least n distinct prime divisors , WLOG , assume A_n is odd(we can easily preclude the power of 2 in prime decompositions of m , thus may assume m is odd) , so $2^{k_n+\varphi(A_n^2)} - m \equiv A_n = 2^{k_n} - m \pmod{A_n^2}$, whence $\frac{2^{k_n+\varphi(A_n^2)}-m}{A_n} \equiv 1 \pmod{A_n}$ so there exist a prime $p \nmid A_n$ but divides $\frac{2^{k_n+\varphi(A_n^2)}-m}{A_n}$, so we find at least $n+1$ distinct prime divisors.

Problem-11. Let ≥ 3 , being prime and for positive integers $a_1, a_2, \ldots, a_{p-2}$. we know that neither a_k , $a_k^k - 1$ is not divisible by p where $k = 1,2,..,p-2$. Prove that there is a positive integer m, such that $a_1 a_2 \ldots a_m \equiv 2$ (mod p).

Bulgarian Olympiads

Solution: we prove inductively , that for any $i = 2,3,\ldots,p-1$ there exist a sequence b_i which are the product of some of $a_1, a_2, \ldots, a_{p-2}$ such that for all $m \neq n$ we have $b_m \not\equiv b_n \pmod{p}$. for $i = 2$, take $b_1 = 1$, $b_2 = a_1$. And we are done . assume there are $b_1, \ldots, b_i$ such that for all $m \neq n$ we have $b_m \not\equiv b_n \pmod{p}$. consider numbers $b_1 a_i, \ldots, b_i a_i$ by induction step none of which are congruent modulo p m now if there are an index l such that $b_j a_i \equiv b_l \pmod{p}$ for any j then numbers $b_1 a_i, \ldots, b_i a_i$ and $b_1, \ldots, b_i$ are same modulo p , comparing theirs product we have ${a_i}^i \equiv 1 \pmod{p}$, a contradiction. Thus we can adjoin new element to the set of incongruencies. Then we can say that for all $t = 2, \ldots, p-1$ there are some product of a_i s . such that is congruent to t modulo p.

Strong Induction and Number theory

Problem-1. Prove that for all positive integer , there exist an integer n , such that $\varphi(n) = m!$

Solution: we know that $3! = \varphi(18)$, $4! = \varphi(72)$, now take an induction , set $p_n \geq 5$, $n-\text{th}$ prime number we know that for $k < p_n$, $k!$ wasn't divisible by p_n. We know that $(p_n - 2)! = \varphi(p_1^{e_1}.p_2^{e_2} \ldots . p_{n-1}^{e_{n-1}})$, now $p_n! = p_n(p_n-1).(p_n-2)! = \varphi(p_n^2).\varphi(p_1^{e_1}.p_2^{e_2}\ldots.p_{n-1}^{e_{n-1}}) = \varphi(p_1^{e_1}.p_2^{e_2}\ldots.p_{n-1}^{e_{n-1}}.p_n^2)$. now for all $p_n < m < p_{n+1} < 2p_n$, if $m = \prod_{p_i < p_n} {p_i}^{\alpha_i}$, then as $(m-1)!$ could be constructed before we can say that $m! = m.(m-1)!$, now as $1 + p_n = p_1^{\beta_1}.p_2^{\beta_2}\ldots.p_{n-1}^{\beta_{n-1}}$, we can construct $(1+p_n)!$ by $(1+p_n).p_n! = p_1^{\beta_1}.p_2^{\beta_2}\ldots.p_{n-1}^{\beta_{n-1}}\varphi(p_1^{e_1}.p_2^{e_2}\ldots.p_{n-1}^{e_{n-1}}.p_n^2) = \varphi(p_1^{e_1+\beta_1}.p_2^{e_2+\beta_2}\ldots.p_{n-1}^{e_{n-1}+\beta_{n-1}}.p_n^2)$ by the same method we can construct all the numbers m, between (p_n, p_{n+1}) so by induction on prime numbers our proof is complete.

Problem-2. Let a, b, m being positive integers such that $\gcd(b,m) = 1$ prove that the set $\{a^n + bn | n = 1,2,\ldots,m^2\}$ contains complete residue system modulo m. **Z.We.Sun**

Solution: we prove the statement of the problem m by induction On m but at first we divide the problem in two cases:

Case-I: $\gcd(m, ab) = 1$, the statement is clear for $m = 1$, assume $m = p_1^{\alpha_1} \ldots . p_t^{\alpha_t}$ where $p_1 < \cdots < p_t$ are distinct prime numbers , set $m_0 = \frac{m}{p_t}$ thus by induction hypothesis there exist a number $k \le m_0^2$, such that for all integer r , we have $a^k + bk = r + m_0 q_0$ for some integer q_0 since $\gcd(\, b(p_1 - 1) \ldots . (p_t - 1), p_t) = 1$ there exist a nonnegative integer $q < p_t$ such that $qb(p_1 - 1) \ldots . (p_t - 1) \equiv -q_0 \pmod{p_t}$.

Set $n = k + m_0 q(p_1 - 1) \ldots . (p_t - 1)$ and since $\varphi(m) |\, m_0(p_1 - 1) \ldots . (p_t - 1)$ then

$$a^n + bn \equiv a^k + b\big(k + m_0 q(p_1 - 1) \ldots . (p_t - 1)\big) \equiv r + m_0\big(q_0 + qb(p_1 - 1) \ldots . (p_t - 1)\big) \equiv r \pmod{m}$$

Now we can see that :

$$0 < k \le n \le m_0^2 + m_0(p_1 - 1) \ldots . (p_t - 1) < m_0^2 + m_0(p_t - 1)m = m_0^2(1 + {p_t}^2 - p_t) < m_0^2 {p_t}^2 = m^2$$

Case-2: $\gcd(m, a) > 1$, set $m = uv$ where $u > 1$ such that a is divisible by any prime divisors of u and $\gcd(v, a) = 1$. set r as arbitrary integer then there exist an integer $s \in \{0,1, \ldots, u - 1\}$ such that $bs \equiv r \pmod{u}$, we also know that $\gcd(a^u, v) = 1$ henceforth $\gcd((a^{-1})^s bu, v) = 1$ whence by previous case , there are $k \le v^2$ such that $(a^u)^k + (a^{-1})^s buk \equiv (a^{-1})^s (r - bs) \pmod{v}$, now set $n = uk + s$ then $n \le uv^2 + u - 1 < u(v^2 + 1) \le u(u^2 + v^2) \le u^2 v^2 = m^2$, and then :

$$a^{uk+s} + buk \equiv r - bs \pmod{v}$$

Implies that $a^n + bn \equiv r (\mathrm{mod}\ v)$, now for any prime $p$divides a we know that $v_p(u) < u$ since $p^u \ge 2^u \ge u + 1$ so $a^n + bn = a^{uk+s} + b(uk + s) \equiv 0 + bs \equiv r \pmod{u}$. and our proof is complete.

Problem-3. Let $a^n - 1$ being divisible by n , prove that $a + 1\, , a^2 + 2\, , \ldots , a^n + n$ is distinct modulo . **Komal**

Solution: we will proceed induction for this , the case $n = 1$ is obvious lets suppose the statement holds true for all integers less than n , now assume $\mathrm{ord}_n^a = k$ then $k|n$ and thus $k < n$, thus the statement of the problem is true for k , hence $+1\, , a^2 + 2\, , \ldots , a^k + k$ are all distinct modulo k , now we prove that for $1 \le x, y \le n$ $a^x + x \not\equiv a^y + y$, let $x = kz + t\, , y = ku + v$ where $0 \le t, v \le k - 1\, , 0 \le z, t \le \frac{n}{k}$ then $a^x \equiv a^t\, , a^y \equiv a^v \pmod{n}$ now we have two cases:

Case-1: $t \ne v$. then $a^x + x \equiv a^t + t \not\equiv a^v + v \equiv a^y + y \pmod{k}$ the central incongruence is derived from induction hypothesis.

Case-2: $= v$, then we essentially have $z \ne u$, so

$$a^x + x \equiv a^t + kz + t = a^v + ku + v + k(z - u) \equiv a^y + y + k(z - u) \not\equiv a^y + y \pmod{n}$$

Our proof is complete.

Problem-4. Prove thet for all positive integer , there exist an integer k , such that $3^{k+1}-2^{k}-k$ is divisible by n. **Serbia**

Solution: define the sequence $x_0=2$, $x_{n+1}=3^{x_n}-2^{x_n}$ we prove the following lemma:

Lemma: for all d the congruence $x_{n+1}\equiv x_n \pmod{d}$ holds for all sufficiently large n.

Proof: we prove this lemma by induction on d , for $d=1$ it is obvious , assume that the statement holds for all integers less than d , since $\varphi(d)<d$ the statement holds for $\varphi(d)$, ensure that $x_{n+1}\equiv x_n \pmod{\varphi(d)}$ for all sufficiently large n , thus $3^{x_{n+1}}\equiv 3^{x_n}$, $2^{x_{n+1}}\equiv 2^{x_n} \pmod{d}$. so our lemma is proven.

Back to our problem , we set $k=x_n$, $d=m$ and we are done ,

$n,\frac{n}{p}$-Technique

Problem-1. We know about the sequence a_n such that $a_1=1$, $a_n=a_{\lfloor\frac{n}{2}\rfloor}+a_{\lfloor\frac{n}{3}\rfloor}+\cdots.+a_{\lfloor\frac{n}{n}\rfloor}+1$, prove that there exist infinitely many n such that $a_n\equiv n$ (mod 2^{2010}) **USA-TST-2010.**

Solution: first we prove the following lemma:

Lemma: if $v_p(n)\geq s$, then $2^{s-1}|a_n-a_{n-1}$.

Proof: assume $a_0=0$, the statement is obvious for $s=1$ .assume that the statement holds true for all integers less than n , we will prove the truth of the proposition for n , assume contrary so n is the least integer such that $v_p(n)\geq s$, but $2^{s-1}\nmid a_n-a_{n-1}$, take $b_n=a_n-a_{n-1}$, then one can find that :

$$b_n=\sum_{i>1,i|n}b_{\frac{n}{i}}=\sum_{i<n,i|n}b_i$$

Thus $b_{\frac{n}{p}}=\sum_{i<\frac{n}{p},i|\frac{n}{p}}b_i$ so $b_n=b_{\frac{n}{p}}+\sum_{i<\frac{n}{p},i|\frac{n}{p}}b_i+\sum_{i<n,i\nmid\frac{n}{p},i|n}b_i=2b_{\frac{n}{p}}+\sum_{i<n,i\nmid\frac{n}{p},i|n}b_i$

Since $\frac{n}{p}<n$ and $p^{s-1}|\frac{n}{p}$ thus by induction hypothesis , we must have $2^{s-2}|b_{\frac{n}{p}}$ so $2^{s-1}|2b_{\frac{n}{p}}$. now those $i<n$ such that $i\nmid\frac{n}{p}$ but $i|n$ are divisible by p^s so by induction hypothesis we must have $2^{s-1}|b_i$, then we must have $2^{s-1}|b_n$. a contradiction , so Our proof is complete.

Let $p_1, \dots, p_m$ be different primes , by Chinese reminder theorem we can find k such that for all$1 \le i \le m$ we have $p_i^s | k+i$ so by our lemma $2^{s-1} | a_{k+i} - a_{k+i-1}$ so $a_k \equiv a_{k+1} \equiv \cdots . \equiv a_{k+m} \equiv N(\text{ say!}) \pmod{2^{s-1}}$ take $> N + 2^{s-1} + 1$, Then since the set $\{k+N , \dots . , k+N+2^{s-1}-1\}$ covers all residues modulo 2^{s-1} , there exist n belongs to the above set , such that $a_n \equiv n \equiv N \pmod{2^{s-1}}$, $n > N$.

Problem-2. For all , prove that there exist an integer n with at least m prime factors(no necessarily distinct) t such that $2^{kn^2} + 3^{kn^2}$ being divisible by n^3. Where k is a positive integer.

Solution: at first we present the following lemma:

Lemma: let $\gcd(a, b) = 1$ being integers and p be an odd prime such that $v_p(a+b) = s \ge 1$, then $v_p(a^p + b^p) = s + 1$.

Proof : set $a + b = x$ is divisible by p, now

$$\frac{a^p+b^p}{a+b} = \frac{(x-b)^p+b^p}{x} = x^{p-1} - bpx^{p-2} + \cdots . - \binom{p}{2} b^{p-2} x + pb^{p-1} \equiv pb^{p-1} \pmod{p^2}$$

Since $v_p(a^p + b^p) = v_p(a+b) + v_p\left(\frac{a^p+b^p}{a+b}\right)$ we are done.

we prove the statement by induction on m , for $m = 0$ take $n_1 = 1$, assume that for $m-1$ there are an integer n_i with at least $m-1$ prime factors such that ${n_i}^3$ divides $2^{k{n_i}^2} + 3^{k{n_i}^2}$, now we divide the problem in two case:

Case-1: there exist a prime p such that $p \nmid n_i$ but $2^{k{n_i}^2} + 3^{k{n_i}^2}$ being divisible by p, since $p | 2^{k{n_i}^2} + 3^{k{n_i}^2}$ by our lemma $p^3 | 2^{kp^2{n_i}^2} + 3^{kp^2{n_i}^2}$ now, take $n_{i+1} = pn_i$, we are done.

Case-2: n_i and $2^{k{n_i}^2} + 3^{k{n_i}^2}$ has same prime factors . we know that ${n_i}^3 \ne 2^{k{n_i}^2} + 3^{k{n_i}^2}$ otherwise $2^{k{n_i}^2} + 3^{k{n_i}^2} > 3^{{n_i}^2} > {n_i}^3$ thus there exist a prime q such that $v_q({n_i}^3) = \alpha$, $v_q\left(2^{k{n_i}^2} + 3^{k{n_i}^2}\right) = \beta$ and $\beta \ge 1 + \alpha$, take $n_{i+1} = qn_i$ so $q^{\alpha+3} | q^{\beta+2} = v_q\left(2^{k{n_{i+1}}^2} + 3^{k{n_{i+1}}^2}\right)$

Problem-3. Prove that for all positive integer , there exist an integer n Which has exactly k prime divisors and $\frac{2^{n^2}+1}{n^3}$ being an integer.

Solution: we first prove two important lemma:

Lemma-1: Let a be an integer , two following statements are equivalent:

a. $v_p(a+1) = s \geq 1$
b. $v_p(a^p+1) = s+1$

Proof: if $v_p(a+1) = s \geq 1$, then $\frac{a^p+1}{a+1} \equiv p \pmod{p^2}$, since $v_p(a^p+1) = v_p(a+1) + v_p\left(\frac{a^p+1}{a+1}\right)$ we are done , on the other hand , since $p|a^p+1$, by fermat's little theorem we find that $a+1$ is divisible by p and by the above proof , we are done.

Lemma-2; Let a be a positive integer , then there exist prime q such that $q|\frac{a^p+1}{a+1}$ but $q \nmid a+1$ except the case $a=2, p=3$.

Proof: assume contrary then all of theirs common primes will appear in theirs greatest prime divisors , since $\gcd\left(\frac{a^p+1}{a+1}, a+1\right) = 1 \text{ or } p$, we find that $\frac{a^p+1}{a+1}$ must be power of p and at the simultaneously with $a+1$, since by the previous lemma , we find that if $p|a+1$, then $v_p\left(\frac{a^p+1}{a+1}\right) = 1$, we find that $\frac{a^p+1}{a+1} = p$ but $\frac{a^p+1}{a+1} > a^2-a+1 > a+1 \geq p$ except the case $a=2, p=3$. Our proof is complete.

Back to our problem , by lemma-1 , we find that , if $p|2^m+1$ then $p^3|2^{mp^2}+1$. now take $n_1 = p_1 = 3$ since $2^9+1 = 513 = 27.19$, take $p_2 = 19$, $n_2 = p_1p_2$,we construct n_i inductively , assume we construct $n_k = p_1p_2 \dots p_k$ by lemma-2 , there are prime p_{k+1} such that $p_{k+1} \nmid 2^{p_1p_2 \cdots p_{k-1}}+1 = a+1$ but $p_{k+1}|2^{p_1p_2 \cdots p_k}+1 = a^{p_k}+1$, now set $n_{k+1} = p_{k+1}n_k$. since $n_k^3|2^{n_k^2}+1|2^{n_k^2 \cdot p_{k+1}^2}+1$ and by lemma-1. $p_{k+1}^3|\ 2^{n_k^2 \cdot p_{k+1}^2}+1$ we find that $n_k^3p_{k+1}^3 = n_{k+1}^3|2^{n_k^2 \cdot p_{k+1}^2}+1$ $=2^{n_{k+1}^2}+1$ .so we are done.

Problem-4. We know about the sequence a_n , that $\sum_{d|n} da_d = k^n$ where k is a positive integer , prove that terms of this sequence are all positive integers a_n is an integer.

Polish training camps.

Solution: we prove the statement by induction on , for $n=1$ it is obvious , assume it was true for all integers less than n , write the equality as $n.a_n + \sum_{d|n\ d<n} da_d = k^n$ we will prove that $n|k^n - \sum_{d|n\ d<n} da_d$ which leads to the

desired conclusion. Assume that $v_p(n) = r$, $n = p^r x$ for some integer x which is not divisible by p we prove that above summation is divisible by p^r, thus by use of induction hypothesis we can write:

$$k^n - \sum_{d|n\ d<n} da_d \equiv k^n - \sum_{d|p^{r-1}x} da_d \equiv k^{p^r x} - k^{p^{r-1}x} \equiv k^{p^{r-1}x}\left(k^{p^{r-1}(p-1)x} - 1\right) \pmod{p^r}$$

Now if $p|k$ we are done , otherwise by Euler's theorem the term in the bracket is divisible by p^r. Since the above proof is indeed for all primes dividing , we are done.

Miscellaneous

Problem-1. For $n = 1,2,3$ we have three type numbers:

Type-1. Zero

Type-2. Geometric progression $1, n+2\ , (n+2)^2, \ldots$

Type-3. Sum of its various numbers.

Prove that every natural number could be represented as the sum of first type , second type and the third type number.

Moskow-2012

Problem-2. Let $0 < x_k \leq \frac{1}{2}$ prove that $(\frac{n}{x_1+x_2+\cdots.+x_n} - 1)^n \leq \left(\frac{1}{x_1} - 1\right) \ldots . (\frac{1}{x_n} - 1)$ **Tuymada-2000**

First Solution: we prove the following lemmaL

Lemma: that if x, y being two positive reals such that $+y \leq 1$, then we have:

$$\left(\frac{1}{x} - 1\right)\left(\frac{1}{y} - 1\right) \geq (\frac{2}{x+y} - 1)^2$$

Proof: rewrite the inequality in the form $\frac{1-x-y}{xy} + 1 \geq \frac{1-x-y}{(\frac{x+y}{2})^2} + 1$ which is true by the AM-GM Inequality $\leq (\frac{x+y}{2})^2$.

now we can prove inductively (on m) that for all $N = 2^m$ real numbers $x_1 , \ldots , x_N$ such that $0 < x_k \leq \frac{1}{2}$, for $1 \leq k \leq N$ we have $(\frac{N}{x_1+x_2+\cdots.+x_N} - 1)^N \leq \left(\frac{1}{x_1} - 1\right) \ldots . (\frac{1}{x_N} - 1)$, now by back-ward induction we prove the inequality for all n , assume ,we want to prove the inequality for $n < N = 2^m$, numbers $x_1 , \ldots , x_n$ $0 < x_k \leq \frac{1}{2}$, and note that $x_1 + \cdots + x_n = nd$,for some positive real d , since the inequality is true for any N , numbers , take specific(Particular) N , numbers as : $x_1 , \ldots , x_n, x_{n+1} = \cdots . = x_N = d$, notice that $x_1 + \cdots + x_N = nd + (N-n)d = Nd$.

so we write the statement as :

$$(\frac{N}{x_1+x_2+\cdots.+x_N}-1)^N \leq \left(\frac{1}{x_1}-1\right)\ldots.(\frac{1}{x_N}-1)$$

Whence the inequality leads to the following statement:

$$(\frac{N}{x_1+x_2+\cdots.+x_N}-1)^N = (\frac{N}{Nd}-1)^N = (\frac{1}{d}-1)^N \leq \left(\frac{1}{x_1}-1\right)\ldots.\left(\frac{1}{x_N}-1\right) =$$

$$\left(\frac{1}{x_1}-1\right)\ldots.\left(\frac{1}{x_n}-1\right).(\frac{1}{d}-1)^{N-n}$$

So one can find that $(\frac{1}{d}-1)^n = (\frac{n}{x_1+x_2+\cdots.+x_n}-1)^n \leq \left(\frac{1}{x_1}-1\right)\ldots.\left(\frac{1}{x_n}-1\right)$.

We are done.

Second Solution: take $x_1+\cdots+x_n = nd$ as past , assume there are to index , say: x_1, x_2 such that $x_1 < d$, $x_2 > d$. now we prove the following lemma:

Lemma: let $x_1 < d$, $x_2 > d$ then : $\left(\frac{1}{x_1}-1\right)\left(\frac{1}{x_2}-1\right) > \left(\frac{1}{d}-1\right)(\frac{1}{x_1+x_2-d})$

Proof: by simplifying both sides we get $(1-x_1-x_2)(d-x_1)(x_2-d) > 0$. and our proof is complete.

Now , if we change the set $\{x_1, \ldots \ldots, x_n\}$ by $\{d, x_1+x_2-d, x_3, \ldots, x_n\}$, the numbers which isn't equal to d , decreased by one , we continue these procedure (which takes finitely) and final all the numbers are equal to d and thus their products is greater than $(\frac{1}{d}-1)^n$.

Comment: the following hard inequality , is indeed the particular case of the above inequality:

Let $a_1, a_2, , .. a_5$ being positive real numbers , prove that:

$$\frac{a_1+a_2}{2}.\frac{a_2+a_3}{2}.\frac{a_3+a_4}{2}.\frac{a_4+a_5}{2}.\frac{a_5+a_1}{2} \leq \frac{a_1+a_2+a_3}{3}.\frac{a_2+a_3+a_4}{2}.\frac{a_3+a_4+a_5}{3}\frac{a_4+a_5+a_1}{3}$$

Just take $= a_1+a_2+a_3+a_4+a_5$, and take $a_k = a_{k-5}$, then $\frac{a_{k+2}+a_{k+3}+a_{k+4}}{a_{k+1}+a_{k+2}} = \frac{S}{a_{k+1}+a_{k+2}}-1$

,hence the problem leads to proving $\left(\frac{S}{a_1+a_2}-1\right)\ldots\left(\frac{S}{a_1+a_5}-1\right) \geq (\frac{3}{2})^5$, we know $\frac{a_k+a_{k+1}}{S}+\frac{a_{k+2}+a_{k+3}}{S} \leq 1$ then:

$$\left(\frac{S}{a_k+a_{k+1}}-1\right)\left(\frac{S}{a_{k+2}+a_{k+3}}-1\right) \geq (\frac{2S}{a_k+a_{k+1}+a_{k+2}+a_{k+3}}-1)^2 = (\frac{2S}{S-a_{k+4}}-1)^2 = (\frac{2S}{S-a_{k-1}}-1)^2$$

Now we must prove $\prod_{i=1}^{5}(\frac{2S}{S-a_i}-1) \geq (\frac{3}{2})^5$, we set $x_k = \frac{S-a_k}{2S} \leq \frac{1}{2}$. And $x_1+\cdots+x_5 = 2$ Implies that:

$$(\frac{5}{x_1+x_2+\cdots.+x_5}-1)^5=(\frac{3}{2})^5 \le \left(\frac{1}{x_1}-1\right)\ldots.(\frac{1}{x_5}-1)$$

Variant:

Let $p=\sqrt[n]{x_1\ldots x_n}$, prove that $\left(1+\frac{1}{x_1(x_1+1)}\right)\ldots..\left(1+\frac{1}{x_n(x_n+1)}\right)\ge(1+\frac{1}{p(1+p)})^n$.

Solution: first we prove the following lemma:

Lemma: Let $p=\sqrt{xy}$ and x,y being positive real numbers then:

$$\left(1+\frac{1}{x(1+x)}\right)\left(1+\frac{1}{y(1+y)}\right)\ge(1++\frac{1}{p(1+p)})^2$$

Solution: the left hand side of the inequality is equal to: $1+\frac{x^2+x+y^2+y}{xy(1+x)(1+y)}=1+\frac{1}{xy}+\frac{x^2-xy+y^2}{xy(1+x)(1+y)}$ now we must prove $\frac{x^2-xy+y^2}{p^2(1+x)(1+y)}\ge\frac{1}{(1+p)^2}$, set $x+y=s$, we must prove $(s^2-3p^2)(p+1)^2\ge(p^2+s+1)p^2$, since $s\ge 2p$ the inequality is proven. Now by induction the statement holds for all powers of 2 , now by back-ward induction we prove the inequality for all n , assume ,we want to prove the inequality for $n<N=2^m$, numbers $x_1,\ldots,x_n$ such that $x_1\ \ldots x_n=p^n$, since the inequality is true for any N , numbers , take specific(Particular) N , numbers as : $x_1,\ldots,x_n,x_{n+1}=\cdots.=x_N=p$, whence $x_1.\ldots.x_N=p^N$, thus :

$$\begin{aligned}&\left(1+\frac{1}{x_1(x_1+1)}\right)\ldots..\left(1+\frac{1}{x_N(x_N+1)}\right)\\&\quad=\left(1+\frac{1}{x_1(x_1+1)}\right)\ldots..\left(1+\frac{1}{x_n(x_n+1)}\right).(1+\frac{1}{p(1+p)})^{N-n}\\&\quad\ge(1+\frac{1}{p(1+p)})^N\end{aligned}$$

So our proof is complete.

Problem-3. Let $a_1,\ldots.,a_n$ being real numbers , prove that two following statements are equivalent:

a. $a_i+a_j\ge 0$, for all $i\ne j$.
b. If $x_1,x_2,\ldots.,x_n$ being non-negative real numbers adding up to one , then :

$$a_1x_1+\cdots+a_nx_n\ge a_1x_1^2+\cdots+a_nx_n^2$$

Chinese Olympiads